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Impatience and dynamic optimal behavior: A bifurcation analysis of the Robinson–Solow–Srinivasan model[☆]

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ABSTRACT

This paper provides a study of optimal economic growth under discounting in discrete time in a two-sector version of the Robinson–Solow–Srinivasan model. It examines how dynamic optimal behavior changes in response to increasing impatience. The optimal policy function is a pan map for high discount factors and a check map for low discount factors. It is shown that the transformation from the pan map to the check map, for the intermediate range of discount factors, can be quite intricate. This is demonstrated by proving the existence of two bifurcation values of the discount factor in the borderline case of the model, which establishes the possibility of reswitching of optimal actions as the discount factor varies.

1. Introduction

How is dynamic optimal behavior of an agent affected when the agent becomes more impatient? The question is central to the study of dynamic optimization models in economics. It is also surprisingly hard to answer. The question belongs to the study of comparative dynamics and is best addressed by the methods of bifurcation analysis, a branch of the mathematical theory of dynamical systems. However, in the mathematical theory, the dynamical system is given by a parametrized family of maps, and forms the primitive concept on which the theory is based. In dynamic optimization models of economics, on the other hand, this parametrized family of maps is a derived concept, obtained by solving an infinite-horizon dynamic optimization problem, for each specification of the parameter(s). It is, in fact, what is called a *parameter-dependent* optimal policy correspondence (OPC). Thus, in order to address the fundamental question posed above, one needs to conduct bifurcation analysis of dynamic optimization models in economics with respect to the discount factor, and in order to do

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this, one needs to obtain an explicit solution of the optimal policy correspondence for each specification of the discount factor.

Let us elaborate further on what all this means. Given a dynamic optimization model,[1] specified by a transition possibility set and a reduced-form utility function, (Ω, u), one needs to explicitly solve for the optimal policy correspondence for every value of the discount factor, $\rho \in (0, 1)$. There are very few dynamic optimization models for which there are explicit solutions of the optimal policy correspondence. The standard aggregative growth model with a log welfare function on consumption and a Cobb–Douglas production function leads to linear optimal policy functions for the entire range of discount factors. The best known instance of an explicit solution of a nonlinear optimal policy function is the Weitzman example, reported in [3], and discussed extensively by McKenzie [4], Benhabib–Nishimura [5] and Mitra–Nishimura [6]. Neither example has the rich structure to allow for the full spectrum of dynamic optimal behavior ranging from convergence to a steady state to persistent cycles and chaos, as the discount factor changes.

In this paper, we deal with a dynamic optimization framework which is a two-sector version of a model due to [7–9], henceforth referred to as the RSS model.[2] Using the method of dynamic programming, we are able to provide an *explicit solution* of the optimal policy correspondence for the full range of ''high'' and ''low'' discount factors. For the high range, the optimal policy correspondence turns out to be a nonlinear optimal policy function, which we call the ''pan map''; for the low range, it turns out to be a nonlinear optimal policy function, which we call the ''check map''; each map is *invariant* for its range of discount factors. Ours is one of the few examples of an explicit solution of the optimal policy correspondence which has the potential to exhibit the full range of dynamic optimal behavior (ranging from simple dynamics like convergence to a steady state or to a period-two cycle to complicated dynamics involving topological and ergodic chaos, as the parameters of the model change), because the optimal policy correspondence, for significant ranges of the discount factor, turns out to be a non-monotonic and a nonlinear optimal policy function (OPF).

Our bifurcation analysis of optimal behavior with respect to changes in the discount factor may be seen as a continuation of the line of research reported in [15,16,6]. The fact that we present an explicit solution of the optimal policy correspondence makes our results sharper, albeit in the context of a more specific two-sector model. Explicit solutions of nonlinear optimal policy functions (such as the tent map) have been obtained, given a discount factor, by *choosing* the transition possibility set and the utility function appropriately (dependent on the given discount factor). Such constructs, though useful in understanding other issues, are ill-suited to conducting bifurcation analysis with respect to the discount factor, given the transition possibility set and the utility function.[3]

Our choice of this highly simplified two-sector model is dictated considerably by the fact that it is tractable for the purpose for which it is being used. The technology in this two-sector model is specified by a pair of parameters (a, d), where a is the labor per unit of capital production in the investment good sector, and d the depreciation rate of capital. Investment is assumed to be irreversible. A consumption good is produced in the other sector, with one unit of capital and one unit of capital needed per unit of consumption good production. Welfare is derived solely from consumption of this good, and is linear in its argument. The objective is to maximize the present value of welfare levels attained over time, where future welfare levels are discounted using a discount factor $\rho \in (0, 1)$. This is a full description of the model, which can be translated to the standard form of (Ω, u, ρ). For the rest of the discussion, we treat (Ω, u) as given and fixed. The parameter which we focus on (for our bifurcation analysis) is the discount factor, ρ, which varies from one infinite-horizon dynamic optimization problem to another.

Our choice of model provides rich dividends. Without placing further restrictions on the parameters (that is, without postulating particular relationships between the fixed parameters (a, d) and the variable parameter of interest, ρ, we are able to specify in considerable detail the nature of the optimal policy correspondence (OPC) for *all* discount factors. This is summarized in Proposition 2 below and, visually, in Fig. 1.[4] It shows that the OPC is actually an OPF for both low and high ranges of the initial stock, where these are respectively given by the intervals $A = [0, \hat{x}]$ and $C = [k, \infty)$. Fig. 1 also shows that in the middle range of initial stocks, the interval $B = (D \cup E)$, optimal actions must lie in the triangular region GG_1M which is easy to specify in terms of the fixed parameters (a, d). Thus, we have a convenient template in terms of which further investigation of the effect of changes in the discount factor on optimal behavior can be conducted.

The result discussed above facilitates our analysis considerably in two ways. First, it is clear that further investigation can focus entirely on the middle range of initial stocks. Second, since we know exactly the optimal policy function in the high and low ranges of initial stocks, the initial stocks in these ranges that lead to optimal actions in the middle range will provide precise information on the initial stocks in the middle range via the functional equation of dynamic programming.

Pursuing this line of analysis in [14], we were able to provide an explicit solution of the optimal policy correspondence as an optimal policy function, which we call the ''pan map'' (the function VGG_1D in Fig. 1, described analytically by g in (10)

1 For a general overview of the subject, see [23], and the earlier texts by Majumdar et al. [1] and Carlson et al. [2], bearing in mind that the last is in continuous time.

2 The interest in the full-fledged multisectoral version of this model, and its generalizations, is derived from studying the problem of optimal choice of techniques in development planning; see [10] who provides a continuous-time analysis under discounting, and [11], who provide a discrete-time analysis in the undiscounted case. For recent work on the two-sector version of the model, see [12–14] and their references.

3 For a full discussion of this point, and for the relevant literature, the reader is referred to [6].

4 We shall return to Fig. 1 in the exposition below, and the reader can disregard at this stage the legends summarizing various values of the discount factor ρ in terms of the parameters of the model.

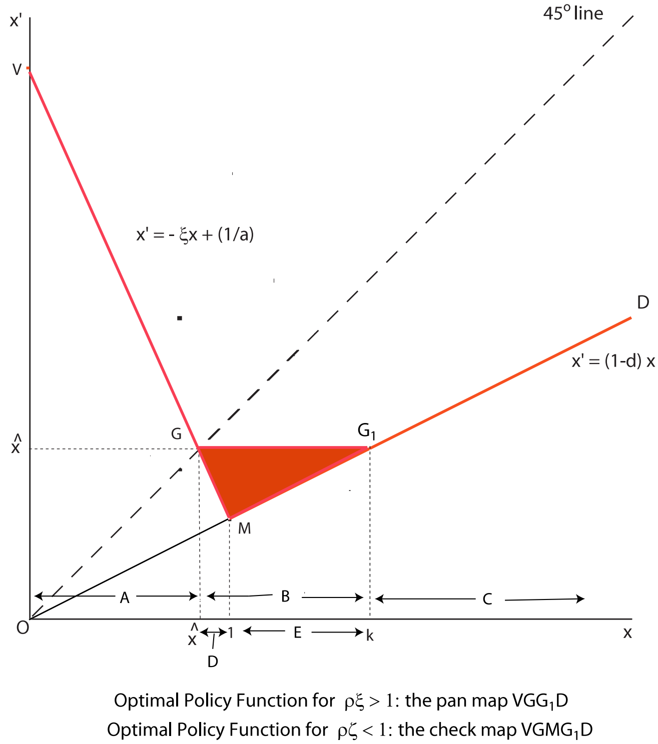

Fig. 1. Optimal policy correspondence for $\rho\xi = 1$: the pan map, the check map and everything in between.

of Section 2.4), for *all* discount factors ρ above a threshold discount factor, $(1/\xi)$, where $\xi \equiv (1/a) - (1-d)$ represents the rate of transformation between machines today and machines tomorrow, while maintaining full-employment of both labor and capital. It turns out that $(1/\xi)$ is in fact a bifurcation value for the discount factor, because once the discount factor falls below it, we can show that the pan map is *not* optimal. [For ease of reference, this result is restated here as Theorem 1.]

The result stated above does not specify the nature of the optimal policy correspondence when the discount factor falls below the threshold level of $(1/\xi)$. Commenting on this aspect of the above result, we conjectured two possible scenarios.[5] The first scenario is that for $\rho < (1/\xi)$, the optimal policy correspondence is an optimal policy function, which we call the "check map" (the function $VGMG_1D$ in Fig. 1, described analytically by H in (CM) of Section 3.1). The second scenario is more intricate. It might be the case that for some values of the discount factor lower than $(1/\xi)$, only a subset of the transitions (in the middle section), described by the correspondence G in Proposition 2, are optimal, and "there is a cascade of bifurcation values" of the discount factor before one reaches the discount factor at which the check map is optimal. This paper may be viewed as an exploration of this conjecture.

Under the technological restriction $\xi(1-d) \leq 1$, the first scenario does indeed occur: for all $\rho < (1/\xi)$, the check map is optimal. [This was shown in [13], and is restated here as Proposition 3.] However, under this technological restriction (generically), the RSS model can generate only simple optimal dynamics: there is a stable period-two cycle, and optimal programs from (almost) every initial stock converge to this period-two cycle.[6]

The second scenario has two aspects. First, there is the assertion that when the discount factor is "well below" the bifurcation value $(1/\xi)$, the check map is always optimal. Second, there is the conjecture that for *some* discount factor below the bifurcation value $(1/\xi)$, the check map is *not* optimal.

The first aspect is relatively straightforward. We show (in Proposition 4) that for all discount factors below the discount factor $(1/\zeta)$, where $\zeta = (1/a)$, the optimal policy correspondence is the check map.

The second aspect is considerably more difficult, and a major part of this paper is devoted to analyzing it. It appears to us that the analysis would have to rely on our separating two cases, one where ξ is relatively low (the "inside case") and the other where it is relatively high (the "outside case"). We can describe the distinction more precisely as follows.[7] Let the parameters $a \in (0, 1)$ and $d \in (0, 1)$ be such that the check map takes the unit capital stock to the stock $k \equiv \hat{x}/(1-d)$ in

5 See the concluding section of [14].

6 See, for example, [17] for a discussion of this point.

7 For a geometric depiction of this categorization, and for a complete analysis of specific instances of the two-sector RSS model within this tripartite categorization, see [18].

two periods; that is, $H^2(1) = k$. This is referred to as the "borderline case"; it forms the borderline between the inside case ($H^2(1) < k$) and the outside case ($H^2(1) > k$).

We approach the second aspect[8] by noting that under the technological restriction $H^2(1) \leq k$ (which is weaker than the technological restriction $\xi(1-d) \leq 1$, discussed above) the check map can be shown to be uniquely optimal if the discount factor ρ is less than $(1/\eta)$, where $\eta = \sqrt{\xi/a}$. [See Proposition 5 in Section 4.2.] This threshold value of the discount factor $(1/\eta)$ is larger than $(1/\zeta)$ and smaller than $(1/\xi)$. Thus, this result improves on Proposition 4, albeit under the technological restriction $H^2(1) \leq k$. But more important for an understanding of the second scenario, the proof of the result suggests that when $(1/\xi) > \rho > (1/\eta)$, the check map might not be optimal (unlike in Proposition 3, where the strong technological restriction $\xi(1-d) \leq 1$ was used). This is explicitly confirmed in Theorem 2, where the technological restriction $H^2(1) = k$ is imposed. To summarize, in the "borderline case", we have the situation that (i) for $\rho < (1/\eta)$, the check map is uniquely optimal, and (ii) for $(1/\xi) > \rho > (1/\eta)$, the check map is *not* optimal. Thus, we have the striking result that for the borderline case, there is a *second* bifurcation value of the discount factor, confirming the conjecture of the second scenario.

Theorem 2 also shows that when $\rho = (1/\xi)$, the check map and the pan map are both optimal (and so is everything in between), and this establishes the possibility of a "reswitching phenomenon". This means that an action that is optimal at the discount factor $(1/\xi)$ can become non-optimal as the discount factor falls to levels in the interval $((1/\eta), (1/\xi))$, and again be optimal as the discount factor falls to yet lower levels in the interval $(0, (1/\eta))$.

In the range of discount factors $(1/\xi) > \rho > (1/\eta)$, neither the check map nor the pan map is optimal in the borderline case. Section 5 is devoted to understanding the nature of the OPC in this situation. It is shown (in Theorem 3) that a flat bottom map (with a narrower bottom than the pan map) is optimal. We note that this flat bottom map is a selection from the optimal policy correspondence (so there is no claim that it is an optimal policy function), and it can vary as the discount factor, ρ, varies in this range. We show (in Proposition 7) that the flat bottom map never gets close to the pan map as ρ varies in this range and it may be speculated that it gradually gets close to the check map as ρ approaches $(1/\eta)$. It is then natural to ask how the flat bottom maps get transformed into the check map as the discount factor, while satisfying the constraints $(1/\xi) > \rho > (1/\eta)$, approaches $(1/\eta)$, the second bifurcation value of the discount factor. In our final result (Proposition 9) we establish the "asymptotic property" that the optimal policy correspondence converges to the check map as the discount factor, ρ, converges to $(1/\eta)$.

The outline of the paper is straightforward. Section 2 presents the preliminaries of the two-sector RSS model. Section 3 provides four lemmas which form our principal tools for characterizing optimality; self-contained proofs of these results are provided in an Appendix. Section 4 presents our main results on bifurcation analysis with respect to the discount factor. Section 5 is devoted exclusively to a more detailed analysis of the optimal policy correspondence in the borderline case.

2. Preliminaries

2.1. The two-sector version of the RSS model

A single consumption good is produced by infinitely divisible labor and machines with the further Leontief specification that a unit of labor and a unit of a machine produce a unit of the consumption good. In the investment-goods sector, only labor is required to produce machines, with $a > 0$ units of labor producing a single machine. Machines depreciate at the rate $0 < d < 1$. A constant amount of labor, normalized to unity, is available in each time period $t \in \mathbb{N}$, where $\mathbb{N}$ is the set of non-negative integers. Thus, in the canonical formulation surveyed in [19], the collection of production plans (x, x'), the amount x' of machines in the next period (tomorrow) from the amount x available in the current period (today), is given by the *transition possibility set*:

$$\Omega = \{(x, x') \in \mathbb{R}^2_+ : x' - (1-d)x \geq 0, \text{ and } a(x' - (1-d)x) \leq 1\}$$

where $z \equiv (x' - (1-d)x)$ is the number of machines that are produced, and $z \geq 0$ and $az \leq 1$ respectively formalize constraints on reversibility of investment and the use of labor. Associated with Ω is the transition correspondence, $\Gamma : \mathbb{R}_+ \to \mathbb{R}_+$, given by $\Gamma(x) = \{x' \in \mathbb{R}_+ : (x, x') \in \Omega\}$. For any $(x, x') \in \Omega$, one can consider the amount y of the machines available for the production of the consumption good, leading to a correspondence $\Lambda : \Omega \longrightarrow \mathbb{R}_+$ with

$$\Lambda(x, x') = \{y \in \mathbb{R}_+ : 0 \leq y \leq x \text{ and } y \leq 1 - a(x' - (1-d)x)\}.$$

Welfare is derived only from the consumption good and is represented by a linear function, normalized so that y units of the consumption good yields a welfare level y. A *reduced-form utility function*, $u : \Omega \to \mathbb{R}_+$ with $u(x, x') = \max\{y \in \Lambda(x, x')\}$ indicates the maximum welfare level that can be obtained today, if one starts with x of machines today, and ends up with x' of machines tomorrow, where $(x, x') \in \Omega$. Intertemporal preferences are represented by the present value of the stream of welfare levels, using a discount factor $\rho \in (0, 1)$.

An *economy* E consists of a triple (a, d, ρ), and the following concepts apply to it. A *program* from x_o is a sequence $\{x(t), y(t)\}$ such that $x(0) = x_o$, and for all $t \in \mathbb{N}$, $(x(t), x(t+1)) \in \Omega$ and $y(t) = \max \Lambda(x(t), x(t+1))$. A *program*

[8] For the motivation for this approach, see the discussion in Section 4.2.

$\{x(t), y(t)\}$ is simply a program from $x(0)$, and associated with it is a *gross investment sequence* $\{z(t+1)\}$, defined by $z(t+1) = (x(t+1) - (1-d)x(t))$ for all $t \in \mathbb{N}$. It is easy to check that every program $\{x(t), y(t)\}$ is bounded by $\max\{x(0), 1/ad\} \equiv M(x(0))$, and in particular every program $\{x(t), y(t)\}$ from $x \in Z \equiv [0, (1/ad)]$ is bounded by $(1/ad)$.

For every program $\{x(t), y(t)\}$, we have $\sum_{t=0}^{\infty} \rho^t u(x(t), x(t+1)) < \infty$. A program $\{\bar{x}(t), \bar{y}(t)\}$ from x_o is called *optimal* if:

$$\sum_{t=0}^{\infty} \rho^t u(x(t), x(t+1)) \leq \sum_{t=0}^{\infty} \rho^t u(\bar{x}(t), \bar{x}(t+1))$$

for every program $\{x(t), y(t)\}$ from x_o. A program $\{x(t), y(t)\}$ is called *stationary* if for all $t \in \mathbb{N}$, we have $(x(t), y(t)) = (x(t+1), y(t+1))$. A *stationary optimal program* is a program that is stationary and optimal; in this case, the stationary level of $x(t)$ is called a *stationary optimal stock*.

The parameter, $\xi = (1/a) - (1-d)$, which figures prominently in our analysis, represents the rate of transformation between machines today and machines tomorrow, while maintaining full-employment of both labor and capital. Without further explicit mention, it will be assumed in what follows that $\xi > 1$. We denote $(1/a)$ by ζ.

2.2. The modified golden rule

A *modified golden rule* is a pair $(\hat{x}, \hat{p}) \in \mathbb{R}^2_+$ such that $(\hat{x}, \hat{x}) \in \Omega$ and:

$$u(\hat{x}, \hat{x}) + (\rho - 1)\hat{p}\hat{x} \geq u(x, x') + \hat{p}(\rho x' - x) \quad \text{for all } (x, x') \in \Omega.$$

Our first proposition establishes the existence of a modified golden rule. A distinctive feature of the RSS model with discounting is that we can describe the modified golden-rule stock explicitly in terms of the parameters of the model.[9]

Proposition 1. *There is* $(\hat{x}, \hat{p}) = (1/(1+ad),\ 1/(1+\rho\xi))$ *such that* $(\hat{x}, \hat{x}) \in \Omega$, *and*

$$u(\hat{x}, \hat{x}) + (\rho - 1)\hat{p}\hat{x} \geq u(x, x') + \hat{p}(\rho x' - x) \quad \text{for all } (x, x') \in \Omega. \tag{MGR}$$

2.3. The dynamic programming approach

In this subsection, we describe the dynamic programming approach to characterizing optimality.[10] Underlying the approach are (a) a value function, and (b) a policy correspondence. Connecting these two objects of interest is the functional equation of dynamic programming.

Using standard methods, one can establish that there exists an optimal program from every $x \in X \equiv [0, \infty)$. Thus, we can define a *value function*, $V : X \to \mathbb{R}$ by:

$$V(x) = \sum_{t=0}^{\infty} \rho^t u(\bar{x}(t), \bar{x}(t+1)) \tag{1}$$

where $\{\bar{x}(t), \bar{y}(t)\}$ is an optimal program from x. Then, it is straightforward to check that V is concave, non-decreasing and continuous on X. Further, it can be verified that V is, in fact, increasing on X.

It can be shown that for each $x \in X$, the Bellman equation:

$$V(x) = \max_{x' \in \Gamma(x)} \{u(x, x') + \rho V(x')\} \tag{2}$$

holds. For each $x \in X$, we denote by $h(x)$ the set of $x' \in \Gamma(x)$ which maximize $\{u(x, x') + \rho V(x')\}$ among all $x' \in \Gamma(x)$. That is, for each $x \in X$,

$$h(x) = \arg\left[\max_{x' \in \Gamma(x)} \{u(x, x') + \rho V(x')\}\right]. \tag{3}$$

Then, a program $\{x(t), y(t)\}$ from $x \in X$ is an optimal program from x if and only if it satisfies the equation: $V(x(t)) = u(x(t), x(t+1)) + \rho V(x(t+1))$ for $t \geq 0$; that is, if and only if $x(t+1) \in h(x(t))$ for $t \geq 0$. We call h the (optimal) *policy correspondence* (OPC). If h is a function, we refer to it as the (optimal) *policy function* (OPF). Standard arguments show that the OPC is upper hemicontinuous on X; consequently, when h is a function, it is continuous on X.

It is easy to verify, using $\rho \in (0, 1)$, that the function V, defined by (1), is the *unique* continuous function on $Z \equiv [0, (1/ad)]$ which satisfies the functional equation of dynamic programming, given by (2).

9 Propositions 1 and 2 are stated and proved in [14].

10 Our exposition is deliberately brief. For a comprehensive accounts of the dynamic programming approach to optimal growth models, see [20,21].

The connection between the value function in the dynamic programming approach and the modified golden rule may be noted as follows.

Proposition 1 shows that

$$(\hat{x}, \hat{p}) = (1/(1 + ad), 1/(1 + \rho\xi)) \tag{4}$$

is a modified golden rule of the economy (Ω, u, ρ). A standard argument can now be used to show that $\hat{x}$ is a stationary optimal stock of the economy (Ω, u, ρ); see, for example, [19, p. 1305]. Consequently, we have:

$$V(\hat{x}) = \hat{x}/(1 - \rho). \tag{5}$$

It is easy to verify that:

$$V(x) - \hat{p}x \le V(\hat{x}) - \hat{p}\hat{x} \quad \text{for all } x \ge 0. \tag{6}$$

Choosing $x = \hat{x} + \varepsilon$ in (6), and letting $\varepsilon \to 0$, we obtain:

$$V'_+(\hat{x}) \le \hat{p} \le V'_-(\hat{x}). \tag{7}$$

Using (7) and (MGR), we have:

$$V'_+(\hat{x}) \le \hat{p} = 1/(1 + \rho\xi) < (a/\rho). \tag{8}$$

2.4. Basic properties of the OPC

In this subsection, we describe basic properties of the optimal policy correspondence, with minimal restrictions on the parameters of our model. These preliminary properties will help us to lay the groundwork for the results to follow. Toward this end, we describe three regions of the state space:

$$A = [0, \hat{x}], \qquad B = (\hat{x}, k), \qquad C = [k, \infty) \tag{9}$$

where $k = \hat{x}/(1 - d)$. In addition, we define a function, $g : X \to X$, by:

$$g(x) = \begin{cases} (1 - d)x & \text{for } x \in C \\ \hat{x} & \text{for } x \in B \\ (1/a) - \xi x & \text{for } x \in A. \end{cases} \tag{10}$$

We refer to g as the "pan map", in view of the fact that its graph resembles a pan.

We further subdivide the region B into two regions as follows:

$$D = (\hat{x}, 1), \qquad E = [1, k) \tag{11}$$

and define a correspondence, $G : X \to X$, by:

$$G(x) = \begin{cases} \{(1 - d)x\} & \text{for } x \in C \\ [(1 - d)x, \hat{x}] & \text{for } x \in E \\ [(1/a) - \xi x, \hat{x}] & \text{for } x \in D \\ \{(1/a) - \xi x\} & \text{for } x \in A. \end{cases} \tag{12}$$

The main result of this subsection can be summarized in the following proposition, and illustrated in Fig. 1.

Proposition 2. *The optimal policy correspondence, h, satisfies:*

$$h(x) \subset \begin{cases} \{g(x)\} & \text{for all } x \in A \cup C \\ G(x) & \text{for all } x \in B. \end{cases} \tag{13}$$

3. Characterizing optimality

In order to conduct a bifurcation analysis of the optimal policy correspondence with respect to the discount factor, it is clear that an explicit solution of the OPC (in terms of the parameters of the model) will be extremely useful. Providing an explicit solution requires us to go beyond the description of the OPC given in Proposition 2; specifically, we need to describe fully the nature of the OPC in the "middle region" of stocks.

This can be accomplished by systematically developing the implications of the functional equation of dynamic programming. The ideas that we pursue in this direction are: (i) the nature of the value function at certain key points in the middle region (specifically, at $\hat{x}$, 1 and k) provides us with valuable information about the OPC in the middle region; and (ii) the nature of the policy function *outside* the middle region (which we already know explicitly) provides us with additional information about the value function in the middle region.

The outcome of the first part of this development is in the form of a set of useful sufficient and necessary conditions of optimality, by using the Kuhn–Tucker method. The second part relates primarily to results on the slope and differentiability of the value function. Together, they provide us with a convenient set of tools to pursue bifurcation analysis in the sections that follow.

3.1. Condition for optimality of the check map

We start with a convenient necessary and sufficient condition for optimality of the check map when the stock x is in the middle region B. For this purpose, it is appropriate at this point to introduce the function H from X to X, defined by:

$$H(x) = \begin{cases} (1-d)x & \text{for } x \in C \cup E \\ (1/a) - \xi x & \text{for } x \in A \cup D. \end{cases} \tag{CM}$$

We refer to H as the "check map" following the terminology of [22], since its graph resembles the standard check mark.

Lemma 1. (I) *Suppose $\bar{x} \in B$, and the following condition holds:*

$$V'_+(H(\bar{x})) \leq (a/\rho).$$

Then,

$$H(\bar{x}) \in h(\bar{x}).$$

Further, if $V'_+(H(\bar{x})) < (a/\rho)$, then $h(\bar{x}) = \{H(\bar{x})\}$.

(II) *Suppose $\bar{x} \in B$, and the following condition holds:*

$$H(\bar{x}) \in h(\bar{x}).$$

Then,

$$V'_+(H(\bar{x})) \leq (a/\rho).$$

3.2. Condition for optimality of a flat bottom map

The check map has a kink at its bottom point, where the stock level is equal to 1. In contrast to that, we might consider maps which have a *flat bottom*. This includes the pan map, of course, but we use the term more generally even when the flat bottom is narrower than in the pan map (where the bottom is the interval $[\hat{x}, k]$).

We now provide a convenient sufficient condition for the optimality of a flat bottom map.

Lemma 2. *Let m, n satisfy $\hat{x} \leq m < 1 < n \leq k$, such that $H(m) = H(n) \equiv \hat{z}$. If $V'_+(\hat{z}) \leq (a/\rho)$ and $V'_-(\hat{z}) \geq (a/\rho)$, then*

$$\hat{z} \in h(x) \quad \text{for all } x \in (m, n).$$

Note in particular (by setting $m = \hat{x}$) that since $V'_+(\hat{x}) < (a/\rho)$ by (8), the pan map is optimal if $V'_-(\hat{x}) \geq (a/\rho)$. Consequently, if the pan map is *not* optimal, then we must have $V'_-(\hat{x}) < (a/\rho)$.

3.3. Slope of the value function

The two lemmas above provide criteria for optimality which are phrased in terms of the slope of the value function. In order to apply them effectively, we need to combine them with knowledge about the slope of the value function at some key points in the state space. Our first result in this direction provides an upper bound on the slope of the value function at $H(1)$.

Lemma 3. *The value function, V, satisfies:*

$$V'_-(H(1)) \leq 1.$$

3.4. Differentiability of the value function

Our second result provides a condition under which the value function is differentiable at $\hat{x}$ and at k.

Lemma 4. (a) *If $V'_-(\hat{x}) < (a/\rho)$, then there is $\theta > 0$, such that* (i) *for all $x \in (k-\theta, k)$, $H(x) \in h(x)$;* (ii) *for all $x \in (\hat{x}, \hat{x}+\theta)$, $H(x) \in h(x)$.*

(b) *If $V'_-(\hat{x}) < (a/\rho)$ and $\rho\xi \neq 1$, then the value function is differentiable at $\hat{x}$ and k, and:*

$$\text{(i) } V'(\hat{x}) = \hat{p} = 1/(1+\rho\xi); \qquad \text{(ii) } V'(k) = \rho(1-d)\hat{p} = \rho(1-d)/(1+\rho\xi).$$

4. Bifurcation analysis with respect to the discount factor

In this section, we begin our bifurcation analysis of the optimal policy correspondence with respect to the discount factor. We start by recalling the result of [14], which shows that $(1/\xi)$ is a bifurcation value for the discount factor, where

$\xi \equiv (1/a) - (1-d)$ represents the rate of transformation between machines today and machines tomorrow, while maintaining full-employment of both labor and capital. Specifically, we are able to provide an explicit solution of the optimal policy correspondence as an optimal policy function, which we call the "pan map", for all discount factors ρ above the discount factor, $(1/\xi)$. Further, when the discount factor ρ falls below $(1/\xi)$, we can show that the pan map is *not* optimal.

Theorem 1. (i) *If $\rho\xi > 1$, the optimal policy correspondence, h, is an optimal policy function, and satisfies $h(x) = \{g(x)\}$ for all $x \in X$.*
(ii) *If $\rho\xi < 1$, then for every $x \in (\hat{x}, k)$, we have $g(x) \notin h(x)$.*
(iii) *If $\rho\xi \geq 1$, the pan map $g(x)$ is optimal; that is $g(x) \in h(x)$ for all $x \in X$.*

Part (i) of the theorem follows from part (i) of Theorem 1 on page 162 and remark (i) on page 167 in [14]. Part (ii) of the theorem follows from part (ii) of Theorem 1 in [14]. Part (iii) follows from part (i) and the upper hemicontinuity of the OPC with respect to the discount factor ρ.

Theorem 1(ii) does not specify the nature of the optimal policy correspondence when $\rho\xi < 1$. Commenting on this aspect of the above result, [14] conjecture two possible scenarios. The first scenario is that for $\rho < (1/\xi)$, the graph of the optimal policy correspondence is the check map. The second scenario is more intricate. It might be the case that for some values of the discount factor lower than $(1/\xi)$, only a subset of the transitions (in the middle section), described by the correspondence G, are optimal, and "there is a cascade of bifurcation values" of the discount factor before one reaches the discount factor at which the check map is optimal. The remainder of this section is an exploration of this conjecture.

4.1. The first scenario

It turns out that, *under a technological restriction* (that is, a restriction on the parameters (a, d)), the first scenario does indeed occur. We state the relevant result here, and we refer the reader to [13] for a proof.

Proposition 3. *Suppose the RSS model (a, d, ρ) satisfies:*

$$\xi(1-d) \leq 1 \tag{14}$$

and:

$$\rho < (1/\xi). \tag{15}$$

Then, its optimal policy correspondence, h, is the function given by the check map H.

Theorem 1, together with Proposition 3, indicate that when the technological restriction (14) holds, then there is a *single* bifurcation value of the discount factor, namely $(1/\xi)$.

4.2. Aspects of the second scenario

The second scenario has two aspects. First, there is the premise that when the discount factor is "well below" the bifurcation value $(1/\xi)$, the check map is always optimal. Second, there is the conjecture that for *some* discount factor below the bifurcation value $(1/\xi)$, the check map is *not* optimal.

We settle the first aspect definitively by showing that for all discount factors below the discount factor $(1/\zeta)$, where $\zeta = (1/a)$, the optimal policy correspondence is the check map.

Proposition 4. *If $\rho < (1/\zeta)$, then the optimal policy correspondence, h, is an optimal policy function, and satisfies $h(x) = \{H(x)\}$ for all $x \in X$.*

Proof. Given Proposition 2, we only need to show that $h(x) = \{H(x)\}$ for all $x \in (\hat{x}, k)$. By Lemma 3, we have $V'_+(H(1)) \leq V'_-(H(1)) \leq 1$. Since $\rho < (1/\zeta) = a$, we must have $V'_+(H(1)) < (a/\rho)$. Since $H(x) \geq H(1)$ for all $x \in (\hat{x}, k)$, the concavity of H yields:

$$V'_+(H(x)) < (a/\rho) \quad \text{for all } x \in (\hat{x}, k).$$

Now, the result follows from Lemma 1(I). □

The second aspect is considerably harder to establish. If there is some discount factor below the bifurcation value $(1/\xi)$, for which the check map is *not* optimal., it is plausible to suppose that there is a "smallest" discount factor with this property. In this case, this "smallest" discount factor would clearly be a *second bifurcation value* of the discount factor. And, unlike the first bifurcation value of $(1/\xi)$, the intuition for a second bifurcation value of the discount factor is not altogether clear. To summarize, we do not have a precise target to aim at.

Our best lead in this direction is to recognize, having established Propositions 3 and 4, that under some technological restriction, weaker than that employed in Proposition 3, it might be possible to show that the check map is optimal under a weaker discount factor restriction than that employed in Proposition 4. That is, we trade off discount factor restriction for technological restriction to obtain a result which sits "in between" Propositions 3 and 4. Compared to Proposition 3, the discount factor restriction would be stronger, but the technological restriction would be weaker. Compared to Proposition 4, which has no separate technological restriction, the discount factor restriction would be weaker, but there would be a technological restriction.

The technological restriction that we introduce at this point depends on the behavior of the second iterate of the stock level $x = 1$, under the check map, H, and it leads to the following nomenclature. If $H^2(1) > k$, we refer to the case as the "outside case", since the second iterate falls outside the region B. If $H^2(1) < k$, we refer to it as the "inside case"; the second iterate falls inside the region B. If $H^2(1) = k$, this forms the borderline between the previous two cases, and we refer to it accordingly as the "borderline case".

In what follows we impose the technological restriction that:

$$H^2(1) \le k. \tag{16}$$

It can be verified (see [17]) that:

$$H^2(1) \le k \Longleftrightarrow \left[\xi - \frac{1}{\xi}\right](1-d) \le 1. \tag{17}$$

Thus, condition (14) implies $H^2(1) < k$, and is stronger than it.

Under the restriction (16), we can establish the result that when the discount factor ρ is smaller than $(1/\eta)$, where $\eta \equiv \sqrt{(\xi/a)}$, the optimal policy correspondence is given by the check map, H. Note that since $\xi = (1/a) - (1-d)$, we have (i) $\xi < (1/a)$, so that $(a/\xi) > a^2$, and $(1/\eta) > a = (1/\zeta)$; (ii) $a\xi < 1$, so that $(a/\xi) < (1/\xi^2)$, and $(1/\eta) < (1/\xi)$.

Proposition 5. *Suppose $H^2(1) \le k$, and $\rho < (1/\eta)$. Then $h(x) = \{H(x)\}$ for all $x \in X$.*

Proof. Let $x = H(1) \equiv (1-d)$ and $z = H(x)$. Then $z \in h(x)$ by Proposition 2, and $u(x, z) = x$, so that: and so, using the knowledge of the policy function for low stocks, we can write

$$V(x) = u(x, z) + \rho V(z) = (1-d) + \rho V(H^2(1)). \tag{18}$$

Pick $0 < \varepsilon < \hat{x} - (1-d)$. Then, $x' \equiv x + \varepsilon = (1-d) + \varepsilon \in (0, \hat{x})$, and so $z' \equiv H(x') \in h(x')$ by Proposition 2, and $u(x', z') = (1-d) + \varepsilon$. Thus,

$$V(x+\varepsilon) = u(x+\varepsilon, z') + \rho V(z') = [(1-d)+\varepsilon] + \rho V(H^2(1) - \xi\varepsilon). \tag{19}$$

Using (18) and (19), we get:

$$\frac{V(x+\varepsilon) - V(x)}{\varepsilon} = 1 - \rho\xi \frac{[V(z) - V(z - \xi\varepsilon)]}{\xi\varepsilon}.$$

On letting $\varepsilon \to 0$, we obtain

$$V'_+(H(1)) = 1 - \rho\xi V'_-(z) \le 1 - \rho\xi V'_-(k) \tag{20}$$

where the inequality in (20) follows from the facts that $z = H^2(1) \le k$, and V is concave.

Since $\rho < (1/\eta) < (1/\xi)$, Lemma 2 and Theorem 1(ii) imply that $V'_-(\hat{x}) < (a/\rho)$. Thus, Lemma 4 applies, and V is differentiable at k, with $V'(k) = \rho(1-d)/(1+\rho\xi)$. Using this information and the hypothesis that $V'_+(H(1)) \ge (a/\rho)$, (20) yields the inequality:

$$V'_+(H(1)) \le 1 - \rho^2\xi(1-d)/(1+\rho\xi). \tag{21}$$

We claim that $V'_+(H(1)) < (a/\rho)$. For if the claim was not valid, (21) would yield:

$$a/\rho \le 1 - \{\rho^2\xi(1-d)/(1+\rho\xi)\}.$$

This can be rewritten as:

$$\rho^3\xi(1-d) - \rho^2\xi - \rho(1-a\xi) + a \le 0. \tag{22}$$

Since $(1-a\xi) = a(1-d)$, (22) can also be written as:

$$(\rho^2\xi - a)(\rho(1-d) - 1) \le 0.$$

Since $\rho \in (0, 1)$, we have $(\rho(1-d) - 1) < 0$. Thus, we must have $\rho^2\xi \ge a$, or $\rho \ge \sqrt{(a/\xi)} = (1/\eta)$. This contradiction establishes our claim. Now, the result follows from Lemma 1(I). □

4.3. The second bifurcation in the borderline case

While Proposition 5 does not confirm the conjecture about the second scenario, its proof deserves close inspection. In honing in on the value $(1/\eta)$ as an important benchmark for the discount factor, ρ, it is extremely tight in all steps, except for (20), where $V'_-(k)$ is used as a *lower bound* for $V'_-(z)$, since $z = H^2(1)$ can be less than k. This suggests, that when $H^2(1) = k$, the inequality in (20) would be superfluous, and the proof would indeed be tight in all steps. Thus, when $H^2(1) = k$, we expect that $\rho > (1/\eta)$ would imply that $V'_+(H(1)) > (a/\rho)$, so that the check map would not be optimal. We confirm this expectation in our next result.

Theorem 2. *Suppose $H^2(1) = k$. Then,*

(i) *If $\rho\eta < 1$, the optimal policy correspondence, h, is an optimal policy function, given by the check map, H.*
(ii) *If $\rho\xi < 1 < \rho\eta$, then we have $H(1) \notin h(1)$, and so the check map is not optimal.*
(iii) *If $\rho\xi = 1$, the optimal policy correspondence, h, satisfies $h(x) = G(x)$ for all $x \in X$.*

Proof. Clearly, part (i) follows directly from Proposition 5. To establish part (ii), note that since $\rho < (1/\xi)$, Lemma 2 and Theorem 1(ii) imply that $V'_-(\hat{x}) < (a/\rho)$. Thus, Lemma 4 applies, and V is differentiable at k, with $V'(k) = \rho(1-d)/(1+\rho\xi)$. Thus, we can follow the steps of the proof of Proposition 5 to obtain (instead of (20)),

$$V'_+(H(1)) = 1 - \rho\xi V'_-(k) = 1 - \{\rho^2\xi(1-d)/(1+\rho\xi)\}. \tag{23}$$

We claim that $V'_+(H(1)) > (a/\rho)$. For if the claim was not valid, (23) would yield:

$$a/\rho \geq 1 - \{\rho^2\xi(1-d)/(1+\rho\xi)\}.$$

This can be rewritten as:

$$\rho^3\xi(1-d) - \rho^2\xi - \rho(1-a\xi) + a \geq 0. \tag{24}$$

Since $(1-a\xi) = a(1-d)$, (24) can also be written as:

$$(\rho^2\xi - a)(\rho(1-d) - 1) \geq 0.$$

Since $\rho \in (0, 1)$, we have $(\rho(1-d) - 1) < 0$. Thus, we must have $\rho^2\xi \leq a$, or $\rho \leq \sqrt{(a/\xi)} = (1/\eta)$. This contradiction establishes our claim. Now, the result follows from Lemma 1(II).

To establish (iii), note first that by Theorem 1(iii), the pan map is optimal; that is $g(x) \in h(x)$ for all $x \in X$. Then, for all $x \in (\hat{x}, k]$, we have:

$$1 - \{a\hat{x} - (1-d)x\} = [1 - a(1-d)]\hat{x} + a(1-d)x < x$$

so that $u(x, \hat{x}) = 1 - \{a\hat{x} - (1-d)x\}$, and:

$$V(x) = 1 - \{a\hat{x} - (1-d)x\} + \rho V(\hat{x}).$$

Also,

$$V(\hat{x}) = 1 - \{a\hat{x} - (1-d)\hat{x}\} + \rho V(\hat{x}).$$

Thus, we get for all $x \in (\hat{x}, k]$,

$$V(x) = a(1-d)x + V(\hat{x}) - a(1-d)\hat{x}$$

and this clearly implies:

$$V'_-(k) = a(1-d). \tag{25}$$

We can now follow the steps of the proof of Proposition 5 to obtain (instead of (20)),

$$V'_+(H(1)) = 1 - \rho\xi V'_-(k) = 1 - a(1-d) \tag{26}$$

the second equality in (26) following from (25) and the fact that $\rho\xi = 1$. Thus, $V'_+(H(1)) = a\xi = (a/\rho)$, and by Lemma 1(I) and the concavity of V, we have $H(x) \in h(x)$ for all $x \in (\hat{x}, k)$, and therefore $H(x) \in h(x)$ for all $x \in X$. We already know that $g(x) \in h(x)$ for all $x \in X$. Thus, using the convexity of Ω and concavity of u on Ω, it follows that $h(x) = G(x)$ for all $x \in X$. $\square$

Theorem 2 provides an illustration of the following "reswitching phenomenon" in the borderline case, where $H^2(1) = k$. Consider the optimal action when the initial stock is $x = 1$. When $\rho = (1/\xi)$, since the check map is optimal (by part (iii) of the theorem), it is optimal to choose next period's stock (call it z) to be $z = H(1) = (1-d)$. When ρ falls below $(1/\xi)$ but is above $(1/\eta)$, then (by part (ii) of the theorem) it is *not* optimal to choose $z = H(1) = (1-d)$; that is, one must switch to a different optimal action. When ρ falls even further so that it is below $(1/\eta)$, then (by part (ii) of the theorem) it is again optimal to choose $z = H(1) = (1-d)$; that is, one "reswitches" back to the original action.

5. The nature of the OPC in the borderline case

Theorems 1 and 2 are definitive in describing the nature of the OPC for $(1/\xi) < \rho < 1$, and for $0 < \rho < (1/\eta)$ respectively for the borderline case. In this final section, we describe the OPC for $(1/\xi) > \rho > (1/\eta)$ in the borderline case in more detail than is done in Theorem 2.

5.1. The optimality of a flat bottom map

Note that for the range of discount factors $(1/\xi) > \rho > (1/\eta)$ in the borderline case, neither the check map nor the pan map is optimal (by Theorem 1(ii) and Theorem 2(ii) respectively). We show in this case, using Lemma 2, that a flat bottom map (with a narrower bottom than the pan map) is optimal.

Proposition 6. *Suppose $V'_-(\hat{x}) < (a/\rho)$ and $V'_+(1-d) > (a/\rho)$. Then, there exist m, n satisfying $\hat{x} < m < 1 < n < k$, such that:*

$$\left.\begin{array}{ll}\text{(i)} & V'_+(H(x)) \le (a/\rho) \quad \text{for all } x \in (\hat{x}, m] \cup [n, k) \\ \text{(ii)} & V'_+(H(x)) > (a/\rho) \quad \text{for all } x \in (m, n)\end{array}\right\}$$

and:

$$\left.\begin{array}{ll}\text{(i)} & H(m) = H(n) \in h(x) \quad \text{for all } x \in [m, n] \\ \text{(ii)} & H(x) \in h(x) \quad \text{for all } x \in (\hat{x}, m] \cup [n, k)\end{array}\right\}.$$

Further, V is linear on $[m, n]$, with $V'(x) = a(1-d)$ for all $x \in (m, n)$.

Proof. Since V is concave, there is $\varepsilon > 0$, such that for all $x \in (k-\varepsilon, k]$, we have:

$$V'_+(H(x)) < (a/\rho).$$

Thus, using the concavity of V, there is some $n \in (1, k)$ such that:

$$\left.\begin{array}{ll}\text{(i)} & V'_+(H(x)) \le (a/\rho) \quad \text{for all } x \in [n, k) \\ \text{(ii)} & V'_+(H(x)) > (a/\rho) \quad \text{for all } x \in [1, n)\end{array}\right\}. \tag{27}$$

Clearly, there is $m \in (\hat{x}, 1)$ such that $H(n) = H(m)$. So, using (27) and the definition of H, we also have:

$$\left.\begin{array}{ll}\text{(i)} & V'_+(H(x)) \le (a/\rho) \quad \text{for all } x \in (\hat{x}, m] \\ \text{(ii)} & V'_+(H(x)) > (a/\rho) \quad \text{for all } x \in (m, 1)\end{array}\right\}. \tag{28}$$

Using Lemma 2 and (27)(i) we get:

$$H(x) \in h(x) \quad \text{for all } x \in [n, k).$$

Using Lemma 2 and (28)(i) we get:

$$H(x) \in h(x) \quad \text{for all } x \in (\hat{x}, m].$$

It follows from (27)(ii) that:

$$V'_-(H(n)) \ge (a/\rho). \tag{29}$$

Using (27)(i) and (29), Lemma 2 implies that:

$$H(m) = H(n) \in h(x) \quad \text{for all } x \in [m, n] \tag{30}$$

by using the upper hemicontinuity of the optimal policy correspondence.

For any $x' \in (m, n)$, we can pick $\varepsilon > 0$, such that $J = [x' - \varepsilon, x' + \varepsilon] \subset (m, n)$, and for every $x \in J$, we have:

$$\left.\begin{array}{ll}\text{(i)} & \hat{z} > (1-d)x \\ \text{(ii)} & a[\hat{z} - (1-d)x] < 1 \\ \text{(iii)} & 1 - a[\hat{z} - (1-d)x] < x\end{array}\right\}. \tag{31}$$

The verification of (31) follows exactly the steps used in the proof of Lemma 2. Then, denoting $H(m) = H(n)$ by $\hat{z}$, and using (30) and (31),

$$V(x' + \varepsilon) - V(x') = u(x' + \varepsilon, \hat{z}) - u(x', \hat{z}) = a(1-d)\varepsilon.$$

Similarly,

$$V(x') - V(x' - \varepsilon) = u(x', \hat{z}) - u(x' - \varepsilon, \hat{z}) = a(1-d)\varepsilon.$$

Letting $\varepsilon \to 0$, $V'_+(x') = V'_-(x') = a(1-d)$. Thus, we must have:

$$V'(x) = a(1-d) \quad \text{for all } x \in (m, n) \tag{32}$$

and since V is continuous, V must be linear on $[m, n]$. □

Theorem 3. *Suppose $H^2(1) = k$, and $\rho\eta > 1 > \rho\xi$. Then, there exist m, n satisfying $\hat{x} < m < 1 < n < k$, such that:*

$$\left.\begin{array}{ll}\text{(i)} & V'_+(H(x)) \le (a/\rho) \quad \text{for all } x \in (\hat{x}, m] \cup [n, k) \\ \text{(ii)} & V'_+(H(x)) > (a/\rho) \quad \text{for all } x \in (m, n)\end{array}\right\} \tag{33}$$

and:

$$\left.\begin{array}{ll}\text{(i)} & H(m) = H(n) \in h(x) \quad \text{for all } x \in [m, n] \\ \text{(ii)} & H(x) \in h(x) \quad \text{for all } x \in (\hat{x}, m] \cup [n, k)\end{array}\right\}.$$

Further, V is linear on $[m, n]$, with $V'(x) = a(1-d)$ for all $x \in (m, n)$.

Proof. Since $\rho\xi < 1$, we have $V'_-(\hat{x}) < (a/\rho)$ by Lemma 2 and Theorem 1(ii). Since $\rho\eta > 1 > \rho\xi$, we also have $V'_+(H(1)) > (a/\rho)$ by Theorem 2(ii) and Lemma 1(I). Then, the result follows directly from Proposition 6. □

It would complete our bifurcation analysis of the borderline case of $H^2(1) = k$ if we could identify the values of m and n in Theorem 3 in terms of the parameters of the model (a, d, ρ). We have not been able to do that yet. However, we can indicate a useful "upper bound" on the flat bottom of the map.

Note that $H^2(1) = k > 1$, and $H^2(k) = \hat{x} < k$. Thus, there is a unique $\bar{n} \in (1, k)$, such that $H^2(\bar{n}) = \bar{n}$. Denoting $H(\bar{n})$ by $\hat{z}$, we note that $\hat{z} \in ((1-d), \hat{x})$. Since $H(1) = (1-d)$ and $H(\hat{x}) = \hat{x}$, there is a unique $\bar{m} \in (\hat{x}, 1)$ such that $H(\bar{m}) = \hat{z} = H(\bar{n})$.

Proposition 7. *Suppose $H^2(1) = k$, and $\rho\eta > 1 > \rho\xi$. Let m and n be values, satisfying $\hat{x} < m < 1 < n < k$, and the properties mentioned in Theorem 3. Then, we must have $[m, n] \subset [\bar{m}, \bar{n}]$.*

Proof. Suppose this were not true. Then, using the definitions of $m, n, \bar{m}, \bar{n}$, it must be the case that

$$\hat{x} < m < \bar{m} < 1 < \bar{n} < n < k.$$

Choose $x \in (m, \bar{m})$ with x sufficiently close to $\bar{m}$, so that $H^2(x) \in (1, n)$. Since we have

$$H^2(\bar{m}) = H^2(\bar{n}) = \bar{n} \in (1, n),$$

this can be done. Denote $H(x)$ by s; then $s \in ((1-d), \hat{x})$. Choosing $\varepsilon > 0$, such that $s + \varepsilon \in ((1-d), \hat{x})$, we have

$$V(s) = s + \rho V((1/a) - \xi s) \quad \text{and} \quad V(s+\varepsilon) = s + \varepsilon + \rho V((1/a) - \xi(s+\varepsilon)).$$

This yields

$$\frac{V(s+\varepsilon) - V(s)}{\varepsilon} = 1 - \rho\xi \frac{V((1/a) - \xi s) - V((1/a) - \xi(s+\varepsilon))}{\xi\varepsilon}.$$

On letting $\varepsilon \to 0$, we obtain

$$\begin{aligned} V'_+(s) &= 1 - \rho\xi V'((1/a) - \xi s) = 1 - \rho\xi V'(H^2(x)) \\ &= 1 - \rho\xi a(1-d) = 1 - \rho\xi(1 - a\xi), \end{aligned} \tag{34}$$

the third equality following from Theorem 3, and the fact that $H^2(x) \in (1, n)$. By (33)(ii), we obtain $V'_+(s) = V'_+(H(x)) > (a/\rho)$. Using this in (34) yields $(a/\rho) < 1 - \rho\xi(1 - a\xi)$. This inequality can be rewritten as:

$$\rho^2\xi(1 - a\xi) - \rho + a < 0$$

which yields:

$$a(\rho\xi - 1)(\rho(1-d) - 1) < 0.$$

Since $\rho(1-d) - 1 < 0$, we must have $\rho\xi > 1$, a contradiction that establishes the result. □

Remark. Since $\bar{m}$ and $\bar{n}$ are defined independent of ρ, Proposition 7 indicates that for all $(1/\xi) > \rho > (1/\eta)$, the bottom of the optimal policy has to "stay away" uniformly from the pan map, even when ρ approaches $(1/\xi)$. This, of course, is completely consistent with the upper hemicontinuity of the optimal policy correspondence with respect to the parameters of the model, since the check map and the pan map and everything in between are optimal when $\rho\xi = 1$ (by Theorem 2(iii)).

5.2. The OPC for the second bifurcation value

Theorems 1–3 provide us with information about the OPC in the borderline case for all values of the discount factor except for the second bifurcation value $(1/\eta)$. We now show that when $\rho = (1/\eta)$, the OPC is given by the check map.

Proposition 8. *Let $H^2(1) = k$. For $\rho \in (0, 1)$, satisfying $\rho\eta = 1$, the check map is uniquely optimal.*

Proof. We know that for all $\rho \in (0, 1)$, satisfying $\rho\eta < 1$, the check map is uniquely optimal. Consequently, by the upper hemicontinuity of the optimal policy correspondence with respect to the parameter ρ, we know that the check map is optimal when $\rho = (1/\eta)$. That is, the check map belongs to the optimal policy correspondence for $\rho = (1/\eta)$. By Lemma 1(II), we must then have $V'_+(H(1)) \le (a/\rho)$.

If $V'_+(H(1)) < (a/\rho)$, then we have $V'_+(H(x)) < (a/\rho)$ for all $x \in (\hat{x}, k)$. Thus, in this case, by Lemma 1(I), $h(x) = \{H(x)\}$ for every $x \in (\hat{x}, k)$. Thus, we are left with the case in which $V'_+(H(1)) = (a/\rho)$. We claim, in this case, that $h(x) = \{H(x)\}$ for all $x \in (\hat{x}, k)$. Suppose the claim is not true. We discuss the implications in three steps.

Step 1. Since the claim is not true, there is some $\tilde{x} \in (\hat{x}, k)$, and $\hat{z} \in (H(\tilde{x}), \hat{x})$, such that $\hat{z} \in h(\tilde{x})$. Then, since $\tilde{z} \equiv H(\tilde{x}) \in h(\tilde{x})$ as well, we have:

$$\left.\begin{array}{l} V(\tilde{x}) = u(\tilde{x}, \tilde{z}) + \rho V(\tilde{z}) \\ V(\tilde{x}) = u(\tilde{x}, \hat{z}) + \rho V(\hat{z}) \end{array}\right\}. \tag{35}$$

Clearly, $u(\tilde{x}, \hat{z}) = u(\tilde{x}, \tilde{z}) - a(\hat{z} - \tilde{z})$, so that (35) yields $\rho[V(\hat{z}) - V(\tilde{z})] = [u(\tilde{x}, \tilde{z}) - u(\tilde{x}, \hat{z})] = a(\hat{z} - \tilde{z})$, and:

$$[V(\hat{z}) - V(\tilde{z})]/(\hat{z} - \tilde{z}) = (a/\rho).$$

This implies that $V'_-(\hat{z}) \ge (a/\rho)$. But since $V'_+(H(1)) = (a/\rho)$, and $\hat{z} > H(\tilde{x}) \ge H(1)$, we must also have $V'_-(\hat{z}) \le (a/\rho)$, and so $V'_-(\hat{z}) = (a/\rho)$. That is, V must be linear on $[H(1), \hat{z}]$, with slope (a/ρ).

Define $n \in (1, k)$, such that $H(n) = \hat{z}$. Then, we have

$$\left.\begin{array}{lll} \text{(i)} & V'_+(H(x)) \le (a/\rho) & \text{for all } x \in [n, k) \\ \text{(ii)} & V'_+(H(x)) = (a/\rho) & \text{for all } x \in [1, n) \end{array}\right\}. \tag{36}$$

Clearly, there is $m \in (\hat{x}, 1)$ such that $H(n) = H(m)$. So, using (36) and the definition of H, we also have

$$\left.\begin{array}{lll} \text{(i)} & V'_+(H(x)) \le (a/\rho) & \text{for all } x \in (\hat{x}, m] \\ \text{(ii)} & V'_+(H(x)) = (a/\rho) & \text{for all } x \in (m, 1) \end{array}\right\}. \tag{37}$$

Using $H(n) = \hat{z}$, and (36)(i), we have:

$$V'_-(\hat{z}) = (a/\rho), \quad \text{and} \quad V'_+(\hat{z}) \le (a/\rho)$$

so that Lemma 2 implies:

$$\hat{z} = H(m) = H(n) \in h(x) \quad \text{for all } x \in [m, n] \tag{38}$$

by using the upper hemicontinuity of the optimal policy correspondence. Now, following exactly the proof in Proposition 6, we must have:

$$V'(x) = a(1 - d) \quad \text{for all } x \in (m, n), \tag{39}$$

and since V is continuous, V must be linear on $[m, n]$, with slope $a(1 - d)$.

Step 2. Define $f : [0, 1] \to [(1 - d), (1/a)]$ by $f(x) = (1/a) - \xi x$. This function has an inverse. Denoting the inverse by F, we have $F : [(1 - d), (1/a)] \to [0, 1]$ defined by $F(y) = (1/a\xi) - (y/\xi)$. Now, define, for T even,

$$\begin{array}{l} y(0) = 1, \qquad y(r + 1) = F(y(r)) \quad \text{for } r = 0, 1, \ldots, T \\ y(T + 2) = y(T + 1)/(1 - d), \qquad y(T + 3) = F(y(T + 2)). \end{array} \tag{40}$$

It is easy to check that the first line of (40) is well defined, with $y(1), y(3), \ldots, y(T + 1)$ belonging to $((1 - d), \hat{x})$, and $y(2), y(4), \ldots, y(T)$ belonging to $(\hat{x}, 1)$. Clearly, then, $y(T + 2)$ is well defined, with $y(T + 2)$ belonging to $(1, k)$. Since $f(1 - d) = H^2(1) = k$ by assumption (in the borderline case), and $f(\hat{x}) = \hat{x} < 1 < y(T + 2)$, $y(T + 3)$ is well defined and belongs to $((1 - d), \hat{x})$.

Notice that the sequence is well defined for every finite T, T even. The important feature of the sequence is that, since $\xi > 1$, $y(T+1) \to \hat{x}$ as $T \to \infty$, and so $y(T+2) \to k$ as $T \to \infty$, and $y(T+3) \to (1-d) = H(1)$ as $T \to \infty$. Consequently, we can choose T large enough, T even, such that $(1 - d) < y(T + 3) < \hat{z} = H(m) = H(n)$.

Step 3. We now run the sequence $y(r)$ in reverse as follows. Define

$$x(0) = y(T + 3), \qquad x(1) = y(T + 2), \ldots, x(T + 3) = y(0) = 1. \tag{41}$$

Notice that by (40) and (41), we have $x(t+1) = H(x(t))$ for $t = 0, 1, \ldots, T+2$. Furthermore, by the properties checked for the sequence $\{y(r)\}$ following (40), we have $x(0) \in ((1-d), \hat{x})$, $x(1) \in (1, k)$. Further, $x(2), x(4), \ldots, x(T+2)$ belong to $((1-d), \hat{x})$ and $x(3), x(5), \ldots, x(T+1)$ belong to $(\hat{x}, 1)$. By choosing $\varepsilon > 0$ small enough and defining:

$$x'(0) = x(0) + \varepsilon, \qquad x'(t+1) = H(x'(t)) \quad \text{for } t = 0, 1, \ldots, T+2 \tag{42}$$

we can ensure that $x'(0) \in ((1-d), \hat{x})$, $x'(1) \in (1, k)$. Further, $x'(2), x'(4), \ldots, x'(T+2)$ belong to $((1-d), \hat{x})$ and $x'(3), x'(5), \ldots, x'(T+1)$ belong to $(\hat{x}, 1)$. Further, $x'(T+3) > x(T+3) = 1$.

Since the check map is optimal, we must have, for all $t \in \{0, 1, \ldots, T+2\}$,

$$\left.\begin{array}{l} V(x(t)) = u(x(t), x(t+1)) + \rho V(x(t+1)) \\ V(x'(t)) = u(x'(t), x'(t+1)) + \rho V(x'(t+1)) \end{array}\right\}. \tag{43}$$

Using (41)–(43), we can write

$$\begin{aligned} V(x(0)+\varepsilon) - V(x(0)) = {} & \varepsilon + \rho(1-1) - \rho^2(1-d)\xi\varepsilon + \rho^3(1-d)\xi^2\varepsilon - \rho^4(1-d)\xi^3\varepsilon \\ & + \cdots - \rho^{T+2}(1-d)\xi^{T+1}\varepsilon + \rho^{T+3}[V(1+(1-d)\xi^{T+2}\varepsilon) - V(1)]. \end{aligned} \tag{44}$$

This yields

$$\begin{aligned} \frac{V(x(0)+\varepsilon) - V(x(0))}{\varepsilon} = {} & 1 - \rho^2(1-d)\xi + \rho^3(1-d)\xi^2 - \rho^4(1-d)\xi^3 + \cdots - \rho^{T+2}(1-d)\xi^{T+1} \\ & + \rho^{T+3}(1-d)\xi^{T+2}\frac{[V(1+(1-d)\xi^{T+2}\varepsilon) - V(1)]}{(1-d)\xi^{T+2}\varepsilon}. \end{aligned} \tag{45}$$

Thus, letting $\varepsilon \to 0$ in (45), we get:

$$V'_+(x(0)) = 1 - \rho^2(1-d)\xi + \rho^3(1-d)\xi^2 - \rho^4(1-d)\xi^3 + \cdots - \rho^{T+2}(1-d)\xi^{T+1} + \rho^{T+3}(1-d)\xi^{T+2}V'_+(1). \tag{46}$$

To simplify (39), define $\rho\xi = \alpha$, and $S = -\alpha^2 + \alpha^3 - \alpha^4 + \cdots - \alpha^{T+2}$. Then, we obtain

$$S = -\frac{(\alpha^2 + \alpha^{T+3})}{(1+\alpha)}. \tag{47}$$

Using (39) and (47) in (46), we can write

$$\begin{aligned} V'_+(x(0)) &= 1 - \frac{(1-d)}{\xi}\frac{(\alpha^2 + \alpha^{T+3})}{(1+\alpha)} + \rho^{T+3}(1-d)\xi^{T+2}a(1-d) \\ &= 1 - \frac{(1-d)}{\xi}\frac{\alpha^2}{(1+\alpha)} + \rho^{T+3}(1-d)\xi^{T+2}\left\{a(1-d) - \frac{1}{(1+\alpha)}\right\} \\ &= 1 - \frac{(1-d)\rho^2\xi}{(1+\rho\xi)} + \rho^{T+3}(1-d)\xi^{T+2}\left\{a(1-d) - \frac{1}{(1+\rho\xi)}\right\} \\ &= (a/\rho) + \rho^{T+3}(1-d)\xi^{T+2}\left\{a(1-d) - \frac{1}{(1+\rho\xi)}\right\} \end{aligned} \tag{48}$$

the last line of (48) following from the fact that $\rho = (1/\eta) = \sqrt{a/\xi}$.

By (36)(ii) and (37)(ii), we have $V'_+(x(0)) = V'_+(y(T+3)) = (a/\rho)$, since $y(T+3) \in ((1-d), \hat{z})$. Thus, (48) can be true only if $a(1-d) = [1/(1+\rho\xi)]$. But, noting that $a(1-d) = 1 - a\xi$, we then get $(1-a\xi)(1+\rho\xi) = 1$, which yields after simplification $\rho(1-d) = 1$, a contradiction. This establishes our claim that $h(x) = \{H(x)\}$ for all $x \in (\hat{x}, k)$, and hence completes the proof. □

5.3. The asymptotic result

We now look at how the OPC changes as ρ changes, with (a, d) fixed so that $H^2(1) = k$. So, we now need to be careful to recognize that the value function V and the OPC h will, in general, depend on ρ. So, we will use V_ρ and h_ρ to alert us about this dependence.

Let us rewrite Theorem 3 in this new notation. Suppose $H^2(1) = k$, and $\rho \in (0, 1)$ satisfies $\rho\eta > 1 > \rho\xi$. Then, there exist $m(\rho), n(\rho)$ satisfying $\hat{x} < m(\rho) < 1 < n(\rho) < k$, such that

$$\left.\begin{array}{ll} \text{(i)} & H(m(\rho)) = H(n(\rho)) \in h_\rho(x) \quad \text{for all } x \in [m(\rho), n(\rho)] \\ \text{(ii)} & H(x) \in h_\rho(x) \quad \text{for all } x \in (\hat{x}, m(\rho)] \cup [n(\rho), k) \end{array}\right\}. \tag{49}$$

Further, V_ρ is linear on $[m(\rho), n(\rho)]$, with the following properties:

$$V'_\rho(x) = a(1-d) \quad \text{for all } x \in (m(\rho), n(\rho)),$$

$$\left.\begin{array}{ll} \text{(i)} \quad V'_{+\rho}(H(x)) \le (a/\rho) & \text{for all } x \in (\hat{x}, m(\rho)] \cup [n(\rho), k) \\ \text{(ii)} \quad V'_{+\rho}(H(x)) > (a/\rho) & \text{for all } x \in (m(\rho), n(\rho)) \end{array}\right\}. \tag{50}$$

The important restriction to be noted from (49) is that, for all $x \in (\hat{x}, k)$,

$$z \in h_\rho(x) \quad \text{implies} \quad z \ge H(m(\rho)) = H(n(\rho)) > (1-d).$$

We now establish the asymptotic result that the OPC, h_ρ, for $(1/\xi) > \rho > (1/\eta)$, converges to the check map as $\rho \to (1/\eta)$.

Proposition 9. *Suppose $H^2(1) = k$. For $\rho \in (0, 1)$ satisfying $\rho\eta > 1 > \rho\xi$, we have, for each $x \in (\hat{x}, k)$,*

$$z(\rho) \in h_\rho(x) \quad \textit{implies} \quad z(\rho) \to H(x) \quad \textit{as } \rho \to (1/\eta). \tag{51}$$

Proof. Suppose (51) does not hold for some $x \in (\hat{x}, k)$. We fix this x in what follows. Since $z(\rho) \in [\hat{x}, k]$, there must then be a subsequence of ρ (call it ρ^s) converging to $(1/\eta)$, for which $z(\rho^s)$ is convergent, and further the limit of $z(\rho^s)$ is $z > H(x)$. Since $z(\rho^s) \in h_\rho(x)$ for all s, the upper hemicontinuity of h_ρ with respect to ρ implies that $z \in h_{(1/\eta)}(x)$. But, since $z > H(x)$, this contradicts the result proved in Proposition 8. Thus, (51) must hold. □

Appendix

The appendix provides the proofs of Lemmas 1–4, presented in Section 3.

Proof of Lemma 1. (I) Let us define the production correspondence from $\mathbb{R}_+$ to subsets of $\mathbb{R}_+$ by:

$$F(x) = \{z : (x, z) \in \Omega\}.$$

Clearly, for each $x \in \mathbb{R}_+$, the set $F(x)$ is the interval given by:

$$F(x) = [(1-d)x, (1/a) + (1-d)x].$$

We consider two cases separately: (i) $\bar{x} \in E = [1, k)$; (ii) $\bar{x} \in D = (\hat{x}, 1)$. Consider case (i) first. Denote $H(\bar{x})$ by $\bar{z}$, and $\{1 - a[H(\bar{x}) - (1-d)\bar{x}]\}$ by $\bar{y}$. Note that since $\bar{x} \in E$, we have $\bar{y} = 1 \le \bar{x}$, and:

$$u(\bar{x}, \bar{z}) = \min\{\bar{y}, \bar{x}\} = \bar{y} = 1.$$

For $z \in F(\bar{x})$, by feasibility, we have $z \ge H(\bar{x})$, and:

$$u(\bar{x}, z) \le 1 - a[z - (1-d)\bar{x}] = u(\bar{x}, \bar{z}) - a(z - \bar{z}).$$

For $z \in F(\bar{x})$, define the function:

$$W(z) = u(\bar{x}, z) + \rho V(z).$$

Then, by concavity of V, we know that W is concave on $F(\bar{x})$. So,

$$\begin{aligned} W(z) - W(\bar{z}) &= [u(\bar{x}, z) - u(\bar{x}, \bar{z})] + \rho[V(z) - V(\bar{z})] \\ &\le -a(z - \bar{z}) + \rho[V(z) - V(\bar{z})] \\ &\le [-a + \rho V'_+(\bar{z})](z - \bar{z}) \\ &\le 0. \end{aligned}$$

Thus, we have:

$$V(\bar{x}) = \max_{(\bar{x}, z) \in \Omega} [u(\bar{x}, z) + \rho V(z)] = u(\bar{x}, \bar{z}) + \rho V(\bar{z}).$$

This means, by the principle of optimality, that $\bar{z} \in h(\bar{x})$.

Next, consider case (ii). Denote $H(\bar{x})$ by $\bar{z}$, and $\{1 - a[H(\bar{x}) - (1-d)\bar{x}]\}$ by $\bar{y}$. Note that since $\bar{x} \in D$, we have:

$$\begin{aligned} \bar{y} &= 1 - a[(1/a) - \xi\bar{x} - (1-d)\bar{x}] \\ &= a[(1-d)\bar{x} + (1/a)\bar{x} - (1-d)\bar{x}] = \bar{x} \end{aligned}$$

and so:

$$u(\bar{x}, \bar{z}) = \min\{\bar{y}, \bar{x}\} = \bar{x}.$$

For $z \in F(\bar{x})$, by feasibility, we have:

$$u(\bar{x}, z) = \min\{1 - a[z - (1-d)\bar{x}], \bar{x}\}. \tag{A.1}$$

For $z \in F(\bar{x})$, define the function:

$$W(z) = u(\bar{x}, z) + \rho V(z).$$

Now, if $z \in h(\bar{x})$, then $z \geq H(\bar{x})$, and:

$$\begin{aligned} 1 - a[z - (1-d)\bar{x}] &\leq 1 - a[(1/a) - \xi\bar{x} - (1-d)\bar{x}] \\ &= a[(1-d)\bar{x} + (1/a)\bar{x} - (1-d)\bar{x}] = \bar{x} \end{aligned}$$

and so by (A.1), we have:

$$u(\bar{x}, z) = 1 - a[z - (1-d)\bar{x}] = u(\bar{x}, \bar{z}) - a(z - \bar{z}).$$

Then, by concavity of W on $F(\bar{x})$, we get:

$$\begin{aligned} W(z) - W(\bar{z}) &= [u(\bar{x}, z) - u(\bar{x}, \bar{z})] + \rho[V(z) - V(\bar{z})] \\ &= -a(z - \bar{z}) + \rho[V(z) - V(\bar{z})] \\ &\leq [-a + \rho V'_+(\bar{z})](z - \bar{z}) \\ &\leq 0. \end{aligned}$$

Thus, we have:

$$\begin{aligned} V(\bar{x}) &= \max_{(\bar{x},z)\in\Omega} [u(\bar{x}, z) + \rho V(z)] = [u(\bar{x}, z) + \rho V(z) : z \in h(\bar{x})] \\ &\leq u(\bar{x}, \bar{z}) + \rho V(\bar{z}) \leq V(\bar{x}). \end{aligned}$$

This means that the above inequalities must in fact hold as equalities, so that $\bar{z} \in h(\bar{x})$.

If $V'_+(H(\bar{x})) < (a/\rho)$, and $z \in h(\bar{x})$, then strict inequality would hold in (A.1) and (A.3), if $z > \bar{z} = H(\bar{x})$, leading to $W(z) < W(\bar{z})$, a contradiction to the definition of h. Thus, in this case, $h(\bar{x}) = \{H(\bar{x})\}$.

(II) Define $I = \{\varepsilon : 0 \leq \varepsilon \leq (\min\{x, 1\}/a)\}$, $\bar{z} = H(\bar{x})$, $z = \bar{z} + \varepsilon$, and $y = \min\{x, 1\} - a\varepsilon$ for $\varepsilon \in I$. Then $z \geq \bar{z} \geq (1-d)\bar{x}$, and $y + a[z - (1-d)x] = \min\{x, 1\} + a[\bar{z} - (1-d)x] = 1$. Also, $0 \leq y \leq x$. Thus, $(x, y) \in \Omega$ and:

$$\left.\begin{aligned} u(x, z) &= y = \min\{x, 1\} - a\varepsilon \\ V(z) &= V(\bar{z} + \varepsilon) \end{aligned}\right\}.$$

Since $\bar{z} \in h(\bar{x})$, $[u(x, z) + \rho V(z)] = [\min\{x, 1\} - a\varepsilon] + \rho V(\bar{z} + \varepsilon)$ must be maximized at $\varepsilon = 0$ among all $\varepsilon \in I$. The first-order condition for this is:

$$-a + \rho V'_+(\bar{z}) \leq 0$$

which establishes the result. □

Proof of Lemma 2. Let $x \in (m, n)$. We can then verify that:

$$\left.\begin{aligned} &\text{(i)} \quad \hat{z} > (1-d)x \\ &\text{(ii)} \quad a[\hat{z} - (1-d)x] < 1 \\ &\text{(iii)} \quad 1 - a[\hat{z} - (1-d)x] < x \end{aligned}\right\}. \tag{A.2}$$

To verify (A.2)(i), note that $\hat{z} = H(n) = (1-d)n > (1-d)x$. To verify (A.2)(ii), note that:

$$a[\hat{z} - (1-d)x] < a\hat{z} = a(1-d)n \leq a(1-d)k = a\hat{x} < 1.$$

Finally, to verify (A.2)(iii), we consider two cases: (a) $x \geq 1$, (b) $x < 1$. In (a), we have:

$$1 - a[\hat{z} - (1-d)x] < 1 \leq x.$$

In (b), we note that $m < x < 1$, and consequently:

$$\hat{z} = H(m) > H(x) = (1/a) - \xi x$$

so that:

$$a\hat{z} > 1 - a\xi x = 1 - a[(1/a) - (1-d)]x = 1 - x + a(1-d)x.$$

Transposing terms yields the result.

Given (A.2), we can find $\varepsilon > 0$ such that for all $z \in I \equiv (\hat{z} - \varepsilon, \hat{z} + \varepsilon)$, we have:

$$\left.\begin{array}{ll} \text{(i)} & z > (1-d)x \\ \text{(ii)} & a[z - (1-d)x] < 1 \\ \text{(iii)} & 1 - a[z - (1-d)x] < x \end{array}\right\}. \tag{A.3}$$

Define $\phi(x) = \{z : (x, z) \in \Omega\}$, and for $z \in \phi(x)$, define:

$$W(z) = u(x, z) + \rho V(z).$$

Clearly, $\phi(x)$ is a convex set, containing I, and W is concave on its domain.

For $z \in I$, using (A.3), we have $u(x, z) = 1 - a[z - (1-d)x]$, and so:

$$W(z) = 1 - a[z - (1-d)x] + \rho V(z).$$

Since $\hat{z} \in I$, we get:

$$W'_{+}(\hat{z}) = -a + \rho V'_{+}(\hat{z}) \leq 0 \tag{A.4}$$

using $V'_{+}(\hat{z}) \leq (a/\rho)$. Similarly,

$$W'_{-}(\hat{z}) = -a + \rho V'_{-}(\hat{z}) \geq 0 \tag{A.5}$$

using $V'_{-}(\hat{z}) \geq (a/\rho)$.

Now, for all $z \in \phi(x)$, with $z > \hat{z}$, we have:

$$W(z) - W(\hat{z}) \leq W'_{+}(\hat{z})(z - \hat{z}) \leq 0$$

by using the concavity of W on its domain, and (A.4). Similarly, for all $z \in \phi(x)$, with $z < \hat{z}$, we have:

$$W(z) - W(\hat{z}) \leq W'_{-}(\hat{z})(z - \hat{z}) \leq 0$$

by using the concavity of W on its domain, and (A.5). Thus, W is maximized at $\hat{z}$ among all points in $\phi(x)$. That is,

$$\max_{(x,z)\in\Omega} [u(x, z) + \rho V(z)] = u(x, \hat{z}) + \rho V(\hat{z}). \tag{A.6}$$

Since the left-hand side expression in (A.9) is $V(x)$ by the optimality principle, we have:

$$V(x) = u(x, \hat{z}) + \rho V(\hat{z}).$$

This means that $\hat{z} \in h(x)$, as claimed in the lemma. □

Proof of Lemma 3. Let $x = H(1) \equiv (1-d)$. Define $x' = x - \varepsilon$, where $0 < \varepsilon < x$. Note that $z \equiv h(x) = H(x)$ and $u(x, z) = x$. Then $z \geq (1-d)x \geq (1-d)x'$, and:

$$a[z - (1-d)x'] = a[z - (1-d)x] + a(1-d)\varepsilon = 1 - x + a(1-d)\varepsilon < 1 \tag{A.7}$$

so that $(x', z) \in \Omega$. Denote $\tilde{y} = u(x, z) - \varepsilon = x - \varepsilon$. Then, we have $\tilde{y} > 0$, and using (A.7):

$$\tilde{y} + a[z - (1-d)x'] = \tilde{y} + (1-x) + a(1-d)\varepsilon = 1 - \varepsilon[1 - a(1-d)] < 1$$

so that $\tilde{y} \in \Lambda(x', z)$. Thus, we have:

$$\left.\begin{array}{l} V(x) = x + \rho V(z) \\ V(x') \geq \tilde{y} + \rho V(z) = (x - \varepsilon) + \rho V(z) \end{array}\right\}. \tag{A.8}$$

Clearly, (A.8) implies:

$$V(x) - V(x - \varepsilon) \leq \varepsilon$$

and this yields the desired bound on the left-hand derivative of the value function at $x = H(1)$. □

Proof of Lemma 4. We prove (a) as follows. Since $V'_{-}(\hat{x}) < (a/\rho)$, there is $\theta' > 0$ such that for all $z \in (\hat{x} - \theta', \hat{x})$, $V'_{+}(z) < (a/\rho)$. Thus, there is $\theta > 0$, such that for all $x \in (\hat{x}, \hat{x} + \theta)$ and for all $x \in (k - \theta, k)$, we have $V'_{+}(H(x)) < (a/\rho)$. Now, the result follows from Lemma 1.

We prove (b) as follows. Choose $0 < \varepsilon$ such that $\xi^2\varepsilon < \theta$, where θ is given by (a). Then, we have by (a),

$$V(\hat{x}+\varepsilon) = (\hat{x}+\varepsilon) + \rho V(\hat{x}-\xi\varepsilon) \quad \text{and} \quad V(\hat{x}) = \hat{x} + \rho V(\hat{x}).$$

This yields

$$\frac{V(\hat{x}+\varepsilon) - V(\hat{x})}{\varepsilon} = 1 - \rho\xi \frac{V(\hat{x}) - V(\hat{x}-\xi\varepsilon)}{\xi\varepsilon}.$$

On letting $\varepsilon \to 0$, we obtain

$$V'_+(\hat{x}) = 1 - \rho\xi V'_-(\hat{x}). \tag{A.9}$$

By (a), we also have

$$V(\hat{x}+\varepsilon) = (\hat{x}+\varepsilon) + \rho(\hat{x}-\xi\varepsilon) + \rho^2 V(\hat{x}+\xi^2\varepsilon) \quad \text{and} \quad V(\hat{x}) = \hat{x} + \rho\hat{x} + \rho^2 V(\hat{x}).$$

This yields

$$\frac{V(\hat{x}+\varepsilon) - V(\hat{x})}{\varepsilon} = 1 - \rho\xi + \rho^2\xi^2 \frac{V(\hat{x}+\xi^2\varepsilon) - V(\hat{x})}{\xi^2\varepsilon}.$$

On letting $\varepsilon \to 0$, we obtain

$$V'_+(\hat{x}) = 1 - \rho\xi + \rho^2\xi^2 V'_+(\hat{x}).$$

Since $\rho\xi \neq 1$, this implies that $V'_+(\hat{x}) = 1/(1+\rho\xi) = \hat{p}$, and using this information in (A.9), we also have $V'_-(\hat{x}) = 1/(1+\rho\xi) = \hat{p}$.

Choose $0 < \varepsilon < \min\{(k-1), \theta\}$ where θ is given by (a). Then $(k-\varepsilon) > 1$, and we know that $H(k-\varepsilon) \in h(k-\varepsilon)$ by (a). Thus, we can write

$$V(k) = 1 + \rho V(\hat{x}) \quad \text{and} \quad V(k-\varepsilon) = 1 + \rho V(\hat{x} - \varepsilon(1-d)),$$

which yields

$$\frac{V(k) - V(k-\varepsilon)}{\varepsilon} = \rho(1-d)\left[\frac{V(\hat{x}) - V(\hat{x}-\varepsilon(1-d))}{\varepsilon(1-d)}\right].$$

On letting $\varepsilon \to 0$, we obtain

$$V'_-(k) = \rho(1-d)V'_-(\hat{x}) = \rho(1-d)\hat{p}.$$

For $\varepsilon > 0$, we also have $H(k+\varepsilon) \in h(k+\varepsilon)$. So, we can write:

$$V(k) = 1 + \rho V(\hat{x}) \quad \text{and} \quad V(k+\varepsilon) = 1 + \rho V(\hat{x} + \varepsilon(1-d)),$$

which yields

$$\frac{V(k+\varepsilon) - V(k)}{\varepsilon} = \rho(1-d)\left[\frac{V(\hat{x}+\varepsilon(1-d)) - V(\hat{x})}{\varepsilon(1-d)}\right].$$

On letting $\varepsilon \to 0$, we obtain

$$V'_+(k) = \rho(1-d)V'_+(\hat{x}) = \rho(1-d)\hat{p}. \quad \square$$

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